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On Henstock–Kurzweil and McShane integrals of Banach space-valued functions

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Abstract

This paper deals with the relation between the McShane integral and the Henstock–Kurzweil integral for the functions mapping a compact interval $I_0 \subset \mathbb{R}^m$ into a Banach space X and some other questions in connection with the McShane integral and the Henstock–Kurzweil integral of Banach space-valued functions. We prove that if a Banach space-valued function f is Henstock–Kurzweil integrable on I_0 and satisfies Property (P), then I_0 can be written as a countable union of closed sets E_n such that f is McShane integrable on each E_n when X contains no copy of c_0. We further give an answer to the Karták's question.

Keywords: Pettis integral; McShane integral; Henstock–Kurzweil integral

1. Introduction

It is known that the McShane integral and the Henstock–Kurzweil integral are two kinds of the Riemann-type integral. For real-valued functions the McShane integral is equivalent to the Lebesgue integral and the Henstock–Kurzweil integral is equivalent to the Perron integral. R.A. Gordon [7] generalized the definition of the McShane integral for real-valued functions to functions from intervals in $\mathbb{R}$ to Banach spaces and discussed some of its properties. S.S. Cao in [10] defined the Henstock–Kurzweil integral for Banach space-valued functions. It is easy

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to see from the corresponding definitions that for Banach space-valued functions the McShane integrability implies Henstock–Kurzweil integrability.

We are looking into the following problem: Is it true that if a Banach space-valued function f is Henstock–Kurzweil integrable on an interval I_0 then I_0 can be written as a countable union of closed sets E_n such that f is McShane integrable on each E_n?

This is an interesting and unanswered question. In this paper we give under suitable condition over the function f (see Property (P)) an affirmative answer to it when Banach space X contains no copy of c_0.

Furthermore, in his memoir [1] K.M. Ostaszewski mentioned that "the question posed by Karták in [5]—whether, for a Perron-integrable function, one can find a nondegenerate interval on which it is Lebesgue-integrable—remains unanswered." For the one-dimensional real-function's case this is a known result of [14] and for the m-dimensional real-function's case it was answered by Z. Buczolich in [2]. Moreover, Z. Buczolich's method in [2] cannot be used for the Banach space's case. Because the Henstock lemma does not hold in an infinite-dimensional Banach space (see [10]). In this paper, we use the other way to give an answer to the question posed by Karták in [5] for the m-dimensional case of the integrals of Banach-space-valued functions. This is to prove that for a Henstock–Kurzweil integrable function, one can find a nondegenerate interval on which it is McShane integrable when Banach space X contains no copy of c_0. Some other questions in connection with the McShane integral and the Henstock–Kurzweil integral are also studied.

2. Preliminaries

Let I_0 be a compact interval in $\mathbb{R}^m$ (or $\mathbb{R}^1$) and $E \subset \mathbb{R}^m$ (or $\mathbb{R}^1$) a measurable subset of I_0. $\mu(E)$ stands for the Lebesgue measure. The Lebesgue integral of a function f over a set E will be denoted by $(L)\int_E f$. X is a real Banach space with the norm $\|\cdot\|$ and X^* its dual. $B(X^*) = \{x^* \in X^*;\ \|x^*\| \leqslant 1\}$ is the closed unit ball in X^*.

We say that the intervals I and J are nonoverlapping if $\operatorname{int}(I) \cap \operatorname{int}(J) = \emptyset$. By $\operatorname{int} J$ the interior of J is denoted.

A *partial M-partition* D in I_0 is a finite collection of interval-point pairs (I, ξ) with nonoverlapping intervals $I \subset I_0$, $\xi \in I_0$ being the associated point of I. Requiring $\xi \in I$ for the associated point of I we get the concept of a *partial K-partition* D in I_0. We write $D = \{(I, \xi)\}$.

A partial M-partition $D = \{(I, \xi)\}$ in I_0 is an *M-partition* of I_0 if the union of all the intervals I equals I_0 and similarly for a K-partition.

Let δ be a positive function defined on the interval I_0. A partial M-partition (K-partition) $D = \{(I, \xi)\}$ is said to be δ-fine if for each interval-point pair $(I, \xi) \in D$ we have $I \subset B(\xi, \delta(\xi))$ where $B(\xi, \delta(\xi)) = \{t \in \mathbb{R}^m;\ \operatorname{dist}(\xi, t) < \delta(\xi)\}$ and dist is the metric in $\mathbb{R}^m$.

Definition 1. An X-valued function f is said to be *McShane integrable* on I_0 if there exists $S_f \in X$ such that for every $\varepsilon > 0$, there exists $\delta(t) > 0$, $t \in I_0$, such that for every δ-fine M-partition $D = \{(I, \xi)\}$ of I_0, we have

$$\left\| (D)\sum f(\xi)\mu(I) - S_f \right\| < \varepsilon.$$

We write $(M)\int_{I_0} f = S_f$ and S_f is the *McShane integral* of f over I_0.

f is McShane integrable on a set $E \subset I_0$ if the function $f \cdot \chi_E$ is McShane integrable on I_0, where χ_E denotes the characteristic function of E.

We write $(M)\int_E f = (M)\int_{I_0} f\chi_E = F(E)$ for the McShane integral of f on E.

It is well know that the McShane and the Lebesgue integrals are equivalent.

Replacing the term "M-partition" by "K-partition" in the definition above we obtain *Henstock–Kurzweil integrability* and the definition of the *Henstock–Kurzweil integral* $(HK)\int_{I_0} f$.

It is clear that if $f : I_0 \mapsto X$ is McShane integrable, then it is also Henstock–Kurzweil integrable because every K-partition is an M-partition.

The basic properties of the McShane integral and Henstock–Kurzweil integral, for example, linearity, additivity with respect to intervals, etc. can be found in [4–10,12,13,15–21]. We do not present them here. The reader is referred to the above mentioned references for the details.

3. The main results

By Proposition 3.5.4 of [19], the following lemma holds:

Lemma 2. *If $f : I_0 \to X$ is Henstock–Kurzweil (McShane) integrable on I_0, then for each x^* in X^*, $x^*(f)$ is Henstock–Kurzweil (McShane) integrable on I_0 and $(HK)\int_{I_0} x^*(f) = x^*((HK)\int_{I_0} f)$ $((M)\int_{I_0} x^*(f) = x^*((HK)\int_{I_0} f))$.*

In [11] it is shown that if a real-function $f : I_0 \to R$ is Henstock–Kurzweil integrable on I_0, then there exists a sequence of closed sets $F_i \subset I_0$, $i \in \mathbb{N}$, such that $\bigcup_i F_i = I_0$ and f is Lebesgue integrable on each F_i. By Baire theorem, for each perfect set E there is F_{n_0} which contains a portion $P = E \cap I$ of E such that f is McShane integrable on P. So we have

Lemma 3. *If a real-function $f : I_0 \to R$ is Henstock–Kurzweil integrable on I_0, then each perfect set contains a portion on which f is McShane integrable.*

It is also easy to show that the following lemma holds (see [16]):

Lemma 4. *If $f : I_0 \to X$ is Henstock–Kurzweil integrable on I_0, then f is Henstock–Kurzweil integrable on each subinterval I of I_0.*

By Lemmas 2 and 4, we obtain the following Lemma 5.

Lemma 5. *Assume that $f : I_0 \to X$ is Henstock–Kurzweil integrable on I_0 and that for every $x^* \in X^*$ the real function $x^*(f) : I_0 \to \mathbb{R}$ is McShane integrable.*

Then for every interval $I \subset I_0$ we have

$$(M)\int_I x^*(f) = x^*\left((HK)\int_I f\right).$$

Lemma 6. *Assume that $f : I_0 \to X$ is Dunford integrable on I_0 with the indefinite Dunford integral ν defined by*

$$\nu(E) = (D)\int_E f \in X^{**}.$$

Assume that $\nu(J) = (D)\int_J f \in X$ for every interval $J \subset I_0$. Then the following claims are equivalent:

(i) *f is Pettis integrable*;
(ii) *for every sequence $J_i \subset I_0$, $i \in \mathbb{N}$, of nonoverlapping intervals the sum $\sum_{i=1}^{\infty} \nu(J_i)$ is norm convergent in X*;
(iii) *for every $\varepsilon > 0$ there is $\eta > 0$ such that*

$$\|\nu(E)\| = \left\|(D)\int_E f\right\| < \varepsilon$$

provided $E \subset I_0$ is measurable with $\mu(E) < \eta$;
(iv) *ν is countably additive.*

This is Proposition 2B of [6] for the case of $I_0 \subset \mathbb{R}^m$. Note that in [6] D.H. Fremlin and J. Mendoza only proved above lemma for the case of $I_0 \subset \mathbb{R}^1$. In fact, it also holds if the interval I_0 is in $\mathbb{R}^m$.

Lemma 7. *Suppose that the Banach space X contains no copy of the space c_0. Assume that $f : I_0 \to X$ is Henstock–Kurzweil integrable on I_0 and Dunford integrable on I_0 as well.*
Then for every open set $G \subset I_0$ there exists $x_G \in X$ such that

$$(M)\int_G x^*(f) = x^*(x_G)$$

for every $x^* \in X^*$.

Proof. Given λ such that $0 < \lambda < 1$ an interval I in $\mathbb{R}^m$ is called λ-regular if

$$r(I) = \frac{\mu(I)}{[d(I)]^m} > \lambda,$$

($r(I)$ is the regularity of the interval I) and $d(I) = \sup\{|x - y|;\ x, y \in I\}$, $|x - y| = \max\{|x_1 - y_1|, \ldots, |x_m - y_m|\}$, and $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m)$.

Suppose that G is an open subset of I_0.

For $t \in G$ let $\delta(t) > 0$ be such that $B(t, \delta(t)) \subset G$.

Let $0 < \lambda < 1$ be fixed. Define

$$\Phi = \bigl\{I \subset I_0, I \text{ is an interval};\ t \in I \subset B\bigl(t, \delta(t)\bigr),\ r(I) > \lambda,\ t \in G\bigr\}.$$

Then Φ is a Vitali cover of G and if $I \in \Phi$ then $I \subset G$.

By the Vitali covering theorem (see, e.g., [13, Proposition 9.2.4]), there is a sequence E_n, $n \in \mathbb{N}$ (E_n is the finite union of nonoverlapping intervals belonging to Φ), such that

$$\mu(G \setminus E_n) < \frac{1}{n},$$

i.e., $\mu(G \setminus E_n) \to 0$ for $n \to \infty$ and $E_n \subset G$ for any $n \in \mathbb{N}$.

Denote $E_0 = \bigcup_{n=1}^{\infty} E_n$. Since $G \setminus E_0 \subset G \setminus E_n$ for every $n \in \mathbb{N}$ we have $\mu(G \setminus E_0) \leqslant \mu(G \setminus E_n) < \frac{1}{n}$ for every $n \in \mathbb{N}$ and consequently $\mu(G \setminus E_0) = 0$. This yields $\mu(E_0) = \mu(G)$.

Let us set $F_n = \bigcup_{i=1}^{n} E_i$. Then clearly $F_n \nearrow E_0$ for $n \to \infty$ and for every $n \in \mathbb{N}$ the set F_n can be expressed as a finite union of nonoverlapping intervals in $\mathbb{R}^m$.

Set $F_0 = \emptyset$ and define $K_n = F_n \setminus F_{n-1}^o$ where F_{n-1}^o is the interior of the set F_{n-1}. We have $E_0 = \bigcup_{n=1}^{\infty} K_n$, $K_n^o \cap K_l^o = \emptyset$ for $n \neq l$ and again K_n can be expressed as a finite union of nonoverlapping intervals in $\mathbb{R}^m$, i.e.

$$K_n = \bigcup_{i=1}^{p_n} I_i^n,$$

while $\{I_i^n;\ i = 1, \ldots, p_n,\ n \in \mathbb{N}\}$ forms an at most countable system of nonoverlapping intervals contained in E_0.

Since $\bigcup_{n=1}^{p} K_n \subset E_0$, $p \in \mathbb{N}$, we have $\sum_{n=1}^{p} \mu(K_n) = \mu(\bigcup_{n=1}^{p} K_n) \leqslant \mu(E_0) = \mu(G) \leqslant \mu(I_0) < \infty$.

Given $x^* \in X^*$ the real function $x^*(f)$ is McShane integrable on I_0 and therefore it is also Lebesgue integrable on I_0.

Hence the Lebesgue integral $\int_G x^*(f)$ exists and

$$\int_G x^*(f) = \int_{E_0} x^*(f)$$

because $\mu(G \setminus E_0) = 0$ and $E_0 \subset G$.

Further we have

$$\begin{aligned}
(M)\int_G x^*(f) &= (M)\int_{E_0} x^*(f) \\
&= (M)\int_{\bigcup_{n=1}^{\infty} K_n} x^*(f) = (M)\int_{\bigcup_{n=1}^{\infty}\bigcup_{i=1}^{p_n} I_i^n} x^*(f) \\
&= \sum_{n=1}^{\infty}\sum_{i=1}^{p_n}(M)\int_{I_i^n} x^*(f) < \infty.
\end{aligned}$$

Using Lemma 5 we have

$$(M)\int_{I_i^n} x^*(f) = x^*\left((HK)\int_{I_i^n} f\right)$$

for $n \in \mathbb{N}$, $i = 1, \ldots, p_n$, and this gives

$$\sum_{n=1}^{\infty}\sum_{i=1}^{p_n} x^*\left((HK)\int_{I_i^n} f\right) < \infty.$$

It is easy to see that if we take an arbitrary rearrangement of terms in the sum

$$\sum_{n=1}^{\infty}\sum_{i=1}^{p_n}(M)\int_{I_i^n} x^*(f)$$

we obtain the same integral $(M)\int_G x^*(f)$. This means that the series

$$\sum_{n=1}^{\infty}\sum_{i=1}^{p_n} x^*\left((HK)\int_{I_i^n} f\right)$$

of real numbers is unconditionally, and therefore also absolutely, convergent.

Since X contains no copy of c_0, by the Bessaga–Pelczynski theorem [3, p. 22] the series $\sum_{n=1}^{\infty}\sum_{i=1}^{p_n}(HK)\int_{I_i^n} f$ is unconditionally convergent in norm to an element $x_G \in X$.

For every $N \in \mathbb{N}$ we have

$$x^*\left(\sum_{n=1}^{N}\sum_{i=1}^{p_n}(HK)\int_{I_i^n} f\right) = \sum_{n=1}^{N}\sum_{i=1}^{p_n} x^*\left((HK)\int_{I_i^n} f\right)$$

and for $N \to \infty$ the left-hand side of this equality converges to $x^*(x_G)$ while the right-hand side converges to $(M)\int_G x^*(f)$. This yields $(M)\int_G x^*(f) = x^*(x_G)$ for every $x^* \in X^*$ and the proof is complete. □

Lemma 8. *Suppose that the Banach space X contains no copy of the space c_0. Assume that $f: I_0 \to X$ is Henstock–Kurzweil integrable on I_0 and that for every $x^* \in X^*$ the real function $x^*(f): I_0 \to \mathbb{R}$ is McShane integrable.*

Then for every closed set $H \subset I_0$ there exists an element $x_H \in X$ such that

$$(M)\int_H x^*(f) = x^*(x_H)$$

for every $x^* \in X^*$.

Proof. If $H \subset I_0$ is closed then $I_0 \setminus H$ is open and for every $x^* \in X^*$ we have

$$x^*\left((M)\int_{I_0} f\right) = \int_{I_0} x^*(f) = \int_H x^*(f) + \int_{I_0\setminus H} x^*(f) = \int_H x^*(f) + x^*(x_{I_0\setminus H}),$$

where for the open set $I_0 \setminus H$ the element $x_{I_0\setminus H} \in X$ is given by Lemma 7.

Hence

$$\int_H x^*(f) = x^*\left((M)\int_{I_0} f - x_{I_0\setminus H}\right)$$

and we put $x_H = (M)\int_{I_0} f - x_{I_0\setminus H} \in X$. □

The next statement is a corollary of Lemma 6.

Corollary 9. *Suppose that the Banach space X contains no copy of the space c_0. Assume that $f: I_0 \to X$ is Henstock–Kurzweil integrable on I_0 and that for every $x^* \in X^*$ the real function $x^*(f): I_0 \to \mathbb{R}^1$ is McShane integrable. Then for every $\varepsilon > 0$ there is $\eta > 0$ such that*

$$\left\|(D)\int_E f\right\| < \varepsilon$$

provided $E \subset I_0$ is measurable with $\mu(E) < \eta$.

Proof. By Lemma 4, for every interval $J \subset I_0$ we have $\nu(J) = (D)\int_J f = (HK)\int_J f \in X$.

Assume that $J_i \subset I_0$, $i \in \mathbb{N}$, is a sequence of nonoverlapping intervals. Then the Henstock–Kurzweil integral $(HK)\int_{J_i} f \in X$ exists for every $i \in \mathbb{N}$ and by Lemma 5 we have $x^*((HK)\int_{J_i} f) = (M)\int_{J_i} x^*(f)$ for $i \in \mathbb{N}$.

It is easy to see, by McShane integrability of x^*f on I_0, that the series $\sum_{i=1}^{\infty} x^*((HK)\int_{J_i} f)$ of real numbers is absolutely convergent.

Since X contains no copy of c_0, by the Bessaga–Pelczynski theorem [3, p. 22] the series $\sum_{i=1}^{\infty}(HK)\int_{J_i} f = \sum_{i=1}^{\infty}(D)\int_{J_i} f$ is unconditionally convergent in norm to an element $x \in X$.

Then (ii) of Lemma 6 is satisfied and therefore we obtain the corollary. $\square$

Theorem 10. *Suppose that the Banach space X contains no copy of the space c_0. Assume that $f : I_0 \to X$ is Henstock–Kurzweil integrable on I_0 and that f is Dunford integrable on I_0. Then $f : I_0 \to X$ is Pettis integrable.*

Proof. It follows at once from Lemma 8, Corollary 9 and Lemma 6. $\square$

Let us note that if the function $f : I_0 \to X$ is Henstock–Kurzweil integrable, by Lemma 2, for each $x^* \in X^*$ the real function $x^*(f)$ is Henstock–Kurzweil integrable on I_0 and if we further assume that $x^*(f)$ is McShane integrable on a subset H of I_0, then $x^*(f)$ is Henstock–Kurzweil integrable on $G = I_0 \setminus H$ and $(HK)\int_G x^*(f) = (HK)\int_{I_0} x^*(f) - (M)\int_H x^*(f)$.

Now we introduce the following concept:

Property (P). We say that a Henstock–Kurzweil integrable function $f : I_0 \to X$ satisfies the *property* (*P*) if for every open subset G of I_0 there is a family $\{I_n\}$ of nonoverlapping intervals I_n such that $G = \bigcup_{n=1}^{\infty} I_n$, for each $x^* \in X^*$ the function $x^*(f)$ is McShane integrable on $H = I_0 \setminus G$ and equality

$$(HK)\int_G x^*(f) = (HK)\int_{I_0} x^*(f) - (M)\int_H x^*(f) = \sum_{n=1}^{\infty}(HK)\int_{I_n} x^*(f) \tag{1}$$

holds, where the series of the right hand is absolutely convergent.

The following is an example of function defined on an compact interval of $\mathbb{R}^m$, $m > 1$, taking values in a Banach space which contains no copy of c_0 and satisfying Property (P).

Example. Let X be an infinite-dimensional Banach space and contain no copy of c_0. Suppose that a series $\sum_{n=1}^{\infty} x_n$ in X is unconditionally convergent and not absolutely convergent. For every positive integers i, j let

$$I_{i,j} = \left(\frac{1}{i+1}, \frac{1}{i}\right) \times \left(\frac{1}{j+1}, \frac{1}{j}\right) \subset R^2.$$

The set $\{I_{i,j}\}_{i,j=1}^{\infty}$ is countable, we denote its elements by $I_1, I_2, \ldots, I_n, \ldots$ for convenience.

Define a function $f : [0,1] \times [0,1] \to X$ by $f(t) = \frac{1}{\mu(I_n)} x_n$ for t in I_n and $f(t) = 0$ for all other values of t. Obviously, the function f is measurable. By Proposition 2.3.3 and Theorem 6.2.1 of [19], f is Henstock–Kurzweil integrable.

Since X contains no copy of c_0, then for each $x^* \in X^*$, $\sum_{n=1}^{\infty} |x^*(x_n)|$ is convergent. Hence,

$$\int_{I_0} |x^*(f)| = \sum_{n=1}^{\infty} \int_{I_n} |x^*(f)| = \sum_{n=1}^{\infty} |x^*(x_n)| < \infty.$$

It is easy to verify that for every open subset G of I_0 with $G = \bigcup_{k=1}^{\infty} J_k$ and $\{J_k\}$ is a family of nonoverlapping intervals, for each $x^* \in X^*$ the function $x^*(f)$ is McShane integrable on $H = I_0 \setminus G$ and the equality

$$\int_G x^*(f) = \int_{I_0} x^*(f) - (M)\int_H x^*(f) = \sum_{k=1}^{\infty} \int_{J_k} x^*(f)$$

holds. Therefore, f satisfies Property (P).

Lemma 11. *Suppose that the Banach space X contains no copy of the space c_0 and $f : I_0 \to X$ is Henstock–Kurzweil integrable on I_0. Assume that for each $x^* \in X^*$ the function $x^*(f)$ is McShane integrable on a closed subset H of I_0 and f satisfies Property* (P). *Then there exists an element $x_H \in X$ such that*

$$(M)\int_H x^*(f) = x^*(x_H)$$

for each $x^* \in X^*$.

Proof. For each $x^* \in X^*$, the real function $x^*(f)$ is Henstock–Kurzweil integrable on I_0 and by the McShane integrability of $x^*(f)$ on H, $x^*(f)$ is Henstock–Kurzweil integrable on the open set $G = I_0 \setminus H$. It follows from Property (P) that for each $x^* \in X^*$, $\sum_{n=1}^{\infty} (HK)\int_{I_n} x^*(f)$ is absolutely convergent and

$$(HK)\int_G x^*(f) = \sum_{n=1}^{\infty} (HK)\int_{I_n} x^*(f) < \infty. \tag{2}$$

On the other hand, Lemma 4 shows that f is Henstock–Kurzweil integrable on each subinterval I_n, $n \in \mathbb{N}$, and

$$(HK)\int_{I_n} x^*(f) = x^*\left((HK)\int_{I_n} f\right)$$

for each $x^* \in X^*$ and every $n \in \mathbb{N}$.

So, for each $x^* \in X^*$, (2) can be written as follows:

$$(HK)\int_G x^*(f) = \sum_{n=1}^{\infty} (HK)\int_{I_n} x^*(f) = \sum_{n=1}^{\infty} x^*\left((HK)\int_{I_n} f\right) < \infty \tag{3}$$

and $\sum_{n=1}^{\infty} x^*((HK)\int_{I_n} f)$ is absolutely convergent.

Since X contains no copy of c_0, by the Bessaga–Pelczynski theorem [3, p. 22] the series $\sum_{n=1}^{\infty} (HK)\int_{I_n} f$ is unconditionally convergent in norm to an element $x_G \in X$ and

$$(HK)\int_G x^*(f) = \sum_{n=1}^{\infty}(HK)\int_{I_n} x^*(f) = \sum_{n=1}^{\infty} x^*\left((HK)\int_{I_n} f\right)$$
$$= x^*\left(\sum_{n=1}^{\infty}(HK)\int_{I_n} f\right) = x^*(x_G).$$

Hence,

$$(M)\int_H x^*(f) = (HK)\int_{I_0} x^*(f) - (HK)\int_G x^*(f) = x^*\left((HK)\int_{I_0} f - x_G\right).$$

Denote $x_H = (HK)\int_{I_0} f - x_G$, then $x_H \in X$ and $(M)\int_H x^*(f) = x^*(x_H)$. The lemma is proved. $\square$

Remark. For 1-dimensional sense ($I_0 \subset \mathbb{R}^1$), Property (P) in Lemma 11 can be removed. Because if $I_0 \subset \mathbb{R}^1$, $G = I_0 \setminus H$ is an open set in I_0 and further let $\{I_n\}$ be an enumeration of the intervals contiguous to H. Then

$$I_0 = H \cup G = H \cup \left(\bigcup_{n=1}^{\infty} I_n\right) \quad \text{and} \quad \mu(G) = \sum_{n=1}^{\infty} \mu(I_n).$$

Theorem 15.10 of [9] (or Theorem 1 of [18]) shows that

$$(HK)\int_G x^*(f) = (HK)\int_{I_0} x^*(f) - (M)\int_H x^*(f) = \sum_{n=1}^{\infty}(HK)\int_{I_n} x^*(f)$$

and $\sum_{n=1}^{\infty}(HK)\int_{I_n} x^*(f)$ is absolutely convergent. Therefore, Property (P) automatically holds. Hence, we have the following Corollary 12.

Corollary 12. *Suppose that the Banach space X contains no copy of the space c_0 and $f : I_0 \to X$ ($I_0 \subset \mathbb{R}^1$) is Henstock–Kurzweil integrable on I_0. Assume that H is a closed set in I_0 and for each $x^* \in X^*$ the real function $x^*(f)$ is McShane integrable on H. Then there exists an element $x_H \in X$ such that*

$$(M)\int_H x^*(f) = x^*(x_H)$$

for each $x^* \in X^*$.

Remark. In the proof of Lemma 11, (1) is a key form. For the multidimensional Henstock–Kurzweil integral, (1) does not automatically hold, because from the Henstock–Kurzweil integrability of the function f on the interval I_0 one can not deduce that f is Henstock–Kurzweil integrable on a measurable subset of I_0 (see [19]). Therefore, taking notice of Theorem 10, we have the following Corollary 13.

Corollary 13. *Suppose that the Banach space X contains no copy of the space c_0 and H is a closed set in I_0 ($I_0 \subset \mathbb{R}^m$). Assume that f is Henstock–Kurzweil integrable on H and further for* *each $x^* \in X^*$ the real function $x^*(f)$ is McShane integrable on H. Then there exists an element $x_H \in X$ such that*

$$(M)\int_H x^*(f) = x^*(x_H)$$

for each $x^* \in X^*$.

Proof. By Theorem 10 $f\chi_H$ is Pettis integrable on I_0 and therefore f is Pettis integrable on H, so there is an element $x_H \in X$ such that $(M)\int_H x^*(f) = x^*(x_H)$ for each $x^* \in X^*$. $\square$

Theorem 14. *Suppose that the Banach space X contains no copy of the space c_0 and $f : I_0 \to X$ is Henstock–Kurzweil integrable on I_0. Assume that f satisfies Property* (P) *and that f is Dunford integrable on a measurable set $E_0 \subset I_0$.*

Then f is Pettis integrable on E_0.

Proof. The Dunford integrability of f on the measurable set $E_0 \subset I_0$ shows that $x^*(f)$ is McShane integrable on E_0. By Lemma 2, $x^*(f)$ is Henstock–Kurzweil integrable on I_0 for each $x^* \in X^*$. It follows that $x^*(f)$ is Henstock–Kurzweil integrable on $I_0 \setminus E_0$ and

$$(HK)\int_{I_0\setminus E_0} x^*(f) = (HK)\int_{I_0} x^*(f) - (M)\int_{E_0} x^*(f).$$

To prove that f is Pettis integrable on E_0, we have to show that for every measurable $E \subset E_0$ there is $x_E \in X$ such that $(M)\int_E x^*(f) = x^*(x_E)$ for each $x^* \in X^*$.

In fact, for every measurable subset $E \subset E_0$ and for each $x^* \in X^*$, the McShane integrability of $x^*(f)$ on E_0 implies that $x^*(f)$ is McShane integrable on $E \subset E_0$. For each $n \in \mathbb{N}$, there is a sequence of closed subsets $H_n \subset E$ such that

$$H_n \subset H_{n+1}, \qquad \mu(E \setminus H_n) < \frac{1}{n}$$

and

$$\mu\left(E \setminus \bigcup_{n=1}^{\infty} H_n\right) = 0.$$

By the absolutely continuity of the Lebesgue integral we have

$$(M)\int_E x^*(f) = \lim_{n\to\infty}(M)\int_{H_n} x^*(f).$$

It follows from Lemma 11 that there exist $x_{H_n} \in X$, $n = 1, 2, \ldots,$ such that

$$(M)\int_{H_n} x^*(f) = x^*(x_{H_n})$$

and

$$(M)\int_E x^*(f) = \lim_{n\to\infty}(M)\int_{H_n} x^*(f) = \lim_{n\to\infty} x^*(x_{H_n})$$

for each $x^* \in X^*$.

Let $H_0 = \emptyset$. Then for each $x^* \in X^*$,

$$
\begin{aligned}
(M)\int_E x^*(f) &= \lim_{n\to\infty}(M)\int_{H_n} x^*(f) \\
&= \sum_{n=1}^{\infty}\Bigl[(M)\int_{H_n} x^*(f) - (M)\int_{H_{n-1}} x^*(f)\Bigr] \\
&= \sum_{n=1}^{\infty}\bigl[x^*(x_{H_n}) - x^*(x_{H_{n+1}})\bigr] = \sum_{n=1}^{\infty} x^*(x_{H_n} - x_{H_{n+1}}).
\end{aligned}
$$

Since X is contains no copy of c_0, by the Bessaga–Pelczynski theorem [3, p. 22], there is $x_E \in X$ such that $\sum_{n=1}^{\infty}(x_{H_n} - x_{H_{n+1}})$ is unconditionally convergent in norm to $x_E \in X$ and $(M)\int_E x^*(f) = x^*(x_E)$. Hence, f is Pettis integrable on E_0 and the proof is complete. □

In [4, Theorem 8] D.H. Fremlin proved the following result for the case of an interval $I_0 \subset \mathbb{R}^1$.

Lemma 15. *A function $f : I_0 \to X$ is McShane integrable on I_0 if and only if it is Henstock–Kurzweil integrable and Pettis integrable.*

Checking Fremlin's proof it can be seen that it still holds when I_0 is an interval in $\mathbb{R}^m$. In fact, Lemma 15 was also proved in [19] for the case of $I_0 \subset \mathbb{R}^m$.

Remark. The theorem in [20, p. 535] points out that a function $f : I_0 \to X$ is Pettis integrable, then f is Henstock–Kurzweil integrable. In fact, this result is not correct. Otherwise, suppose that the theorem in [20, p. 535] holds, this means that the Pettis integrable function f is Henstock–Kurzweil integrable on I_0. It follows immediately from Lemma 15 that f is McShane integrable on I_0. However, the example 3C [6, p. 143] and the example (CH) [21, p. 1184] show that there is a function f such that f is Pettis integrable but not McShane integrable.

We come now to our main results.

Theorem 16. *Suppose that the Banach space X contains no copy of the space c_0. Assume that the function $f : I_0 \to X$ is Henstock–Kurzweil integrable and satisfies Property* (P). *Then each perfect set contains a portion on which f is McShane integrable.*

Proof. Let E be a perfect set in I_0 and let $\Delta = \{I_n\}$ be the sequence of all open intervals in I_0 that intersect E and have rational endpoints. Let $E_n = E \cap I_n$, $n = 1, 2, \ldots$. For each pair of positive integers m and n let $E_m^n = \{x^* \in X^*\colon \int_{E_n} |x^*(f)| \leqslant m\}$. Then $X^* = \bigcup_m^{\infty}\bigcup_n^{\infty} E_m^n$.

In fact, for each m and n we have $E_m^n \subset X^*$, so $\bigcup_m^{\infty}\bigcup_n^{\infty} E_m^n \subset X^*$. On the other hand, for every $x^* \in X^*$, by Lemma 2, $x^*(f)$ is Henstock integrable on I_0. It follows from Lemma 3 that each perfect set E contains a portion $P = E \cap I$ on which $x^*(f)$ is McShane integrable. So there is a $n_0 \in N$ such that $I_{n_0} \subset I$ and $I_{n_0} \in \Delta$. Note that $x^*(f)$ is McShane integrable on $P = E \cap I$, then $x^*(f)$ is McShane integrable on a portion $E_{n_0} = E \cap I_{n_0} \subset E \cap I$ and therefore there is a m_0 such that $\int_{E_{n_0}} |x^*(f)| \leqslant m_0$. Therefore, $x^* \in E_{m_0}^{n_0}$ and $X^* \supset \bigcup_m^{\infty}\bigcup_n^{\infty} E_m^n$. That is, $X^* = \bigcup_m^{\infty}\bigcup_n^{\infty} E_m^n$.

Now we prove that each of the sets E_m^n is closed.

Let x^* be a limit point of E_m^n and $\{x_k^*\}$ a sequence in E_m^n that converges to x^*. Then the sequence $\{|x_k^* f|\}$ converges pointwise on I_0 to the function $|x^*(f)|$ and by Fatou's lemma we have

$$\int_{E_n} \left|x^*(f)\right| \leqslant \liminf_{k\to\infty}\left\{\int_{E_n} \left|x_k^* f\right|\right\} \leqslant m.$$

This shows that $x^* \in E_m^n$ and conclude that the set E_m^n is closed.

By the Baire Category Theorem, there exist M, N, x_0^*, and $r > 0$ such that

$$\left\{x^*\colon \left\|x^* - x_0^*\right\| \leqslant r\right\} \subset E_M^N.$$

For each x^* in X^* with $\|x^*\| \neq 0$, by $x_0^* \in \{x^*\colon \|x^* - x_0^*\| \leqslant r\}$, $\frac{r}{\|x^*\|}x^* + x_0^* \in \{x^*\colon \|x^* - x_0^*\| \leqslant r\}$, we find that

$$\int_{E_N} \left|x^*(f)\right| \leqslant \frac{\|x^*\|}{r}\left\{\int_{E_N} \left|\frac{r}{\|x^*\|}x^*(f) + x_0^*(f)\right| + \int_{E_N} \left|x_0^*(f)\right|\right\} \leqslant \frac{2M}{r}\|x^*\|.$$

Hence, for each x^* in X^* the function $x^*(f)$ is Lebesgue integrable on the portion $E_N = E \cap I_N$. This shows that f is Dunford integrable on $E \cap I_N$.

According to Theorem 14, we obtain that f is Pettis integrable on $E \cap I_N$. It follows from Lemma 15 that f is McShane integrable on $E \cap I_N = P_0$. □

Note that the "perfect set" in Theorem 16 may be replaced by "closed set", the result still holds.

Remark. In the proof of Theorem 16, taking the perfect set E as a subinterval I of I_0, i.e., $E = I$, the portion $E_N = I \cap I_N$ of I is still a subinterval J of I_0. That is, $J = I \cap I_N$ is a subinterval of I_0. Checking the above proof it can be seen that f is Dunford integrable on the subinterval J. On the other hand, by Lemma 4, f is always Henstock–Kurzweil integrable on the subinterval J of I_0. It follows from Theorem10 that f is Pettis integrable on J. By Lemma 15 f is McShane integrable on J. Hence, we obtain the following theorem.

Theorem 17. *Suppose that the Banach space X contains no copy of the space c_0 and $I_0 \subset \mathbb{R}^m$. If a function $f : I_0 \to X$ is Henstock–Kurzweil integrable, then there exists a subinterval J of I_0 such that f is McShane integrable on J.*

This is an answer to Karták's question from [1] for the Banach space case mentioned in the introduction.

Using the Baire Category Theorem and Theorem 9 in [17] (or Theorem 4.16 in [19]), the following theorem can be obtained.

Theorem 18. *Suppose that the Banach space X contains no copy of the space c_0 and that a function $f : I_0 \to X$ is given. Then I_0 can be written as a countable union of closed sets E_n such that f is McShane integrable on each E_n if and only if every closed set contains a portion on which f is McShane integrable.*

Combining Theorems 16 and 18, we obtain the statement as follows.

Theorem 19. *Suppose that X contains no copy of the space c_0. Assume that a function $f : I_0 \to X$ is Henstock–Kurzweil integrable and satisfies Property* (P). *Then I_0 can be written as a countable union of closed sets E_n such that f is McShane integrable on each E_n.*

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